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Randomly incomplete spectra and intermediate statistics

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By randomly removing a fraction of levels from a given spectrum a model is constructed that describes a crossover from this spectrum to a Poisson spectrum. The formalism is applied to the transitions towards Poisson from random matrix theory (RMT) spectra and picket fence spectra. It is shown that the Fredholm determinant formalism of RMT extends naturally to describe incomplete RMT spectra.

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It is by now well established that, in the semiclassical limit, spectral properties of physical systems whose underlying classical motion is chaotic are consistent with the predictions of the Wigner-Dyson random matrix theory [1], while those with an underlying regular motion behave as an uncorrelated sequence of levels (Poisson statistics) [2] (see Ref. [3] for a review). Matrix models have been proposed to describe transitions between these two extreme behaviors. In these interpolating matrix models the transition is generated by reducing the magnitude of the off-diagonal elements with respect to the diagonal ones. This can be achieved in different ways, for instance by decreasing the variance of all off-diagonal elements [4], by making the matrices banded [5] or by introducing a power-law decay [6] or more complicated schemes [7]. In these kinds of models the invariance of the ensemble probability distribution under unitary transformation is broken. Spectral distributions of some invariant ensembles may also show deviation from the Wigner-Dyson statistics but without reaching the Poisson limit [8,9]. Apparently symmetry breaking is necessary to completely eliminate the correlations among levels [10].

Despite that classical analogy does not support the assumption of a unique crossover from chaotic to regular regimes, the existence of a third universal statistics has been conjectured motivated by the Anderson model description of the metal-insulator transition (MIT) [11]. In the MIT, localized states of the insulator phase obey Poisson statistics while the extended states of the metal phase show a Wigner-Dyson behavior. The third universal behavior would correspond to statistics observed at the critical point. The main characteristics of this behavior would be the multifractality of the eigenstates, nearest neighbor spacing distribution (NND) exhibiting a linear level repulsion with a slope at the origin steeper than the Wigner case and with an exponential decay for large separations in contrast to the Gaussian standard decay. Long range statistics like number variance would increase linearly as in the Poisson case but with a smaller slope [12]. Although ensembles have been found, invariant or not, that show some of these properties, the extent of their validity has not yet been established.

Perhaps the simplest model with these spectral features is the short range plasma model [13]. It consists in a one-dimensional (1D) Coulomb gas model for which the range of the interaction is restricted to a finite number of neighboring levels. An appealing feature of the model is its amenability to analytical treatment. It was found that the intermediate statistics obtained when only adjacent pairs interact, denoted by the authors as the semi-Poisson statistics, gives an approximately good description of spectral properties of some diffractive billiards [14].

It has been shown that semi-Poisson statistics can also be obtained in a completely independent way as a particular case of a family of statistics termed by their authors the daisy models [15]. It consists of removing every other r level from an uncorrelated spectrum. The particular case $r=1$ (every other level) corresponds exactly to the semi-Poisson statistics. Notice also that it has been known for quite a long time that spectral properties of two different classes of spectra can be related by the operation of dropping levels from one of them, namely measures of the symplectic (GSE) and the orthogonal (GOE) ensembles of random matrix theory (RMT) [16] can be connected in this way.

In the above cases, levels are removed in a correlated way (every other level) and the operation results in a more correlated spectrum (from GOE to GSE, from Poisson to semi-Poisson). Our purpose here is to investigate the same operation but performed randomly. Specifically, we consider an infinite spectrum and, after dropping at random a fraction 1-f of levels, the remaining fraction f is studied $(0<f<1)$. In order to keep to unity the average level density of the remaining spectrum is correspondingly contracted. To determine the statistical properties of the transformed spectra we resort to the formalism developed in our previous work [17] in which the problem of randomly incomplete spectra was considered. We showed that when $f\rightarrow 0$, irrespective of the nature of the initial spectrum, the statistical properties approach those of an uncorrelated spectrum (the Poissonian spectrum is a fixed point of the operation). Therefore, the random dropping operation generates models whose statistics are intermediate between those of the initial spectrum and Poisson statistics. Obviously it can also be seen that the statistical properties of a Poisson sequence are not affected by this dropping operation.

In the present communication we discuss properties of models constructed starting with RMT spectra and with a picket fence of equally spaced levels. We show that in the first case a family of intermediate statistics parametrized by the fraction f of the remaining levels is generated that shows features similar to those of the intermediate critical statistics. In particular, the $f=1/2$ case is compared with the semi-Poisson statistics. In the picket fence model we show that the

$f=1/2$ member of the family reproduces the spectral properties of a sequence of levels weakly confined by a logarithm-normal potential (see below).

For later use, we recall some notations and results of Ref. [17]. We consider a spectrum $\rho(E)=\Sigma\delta(E-E_i)$ with mean spacing $\langle\rho\rangle=1$. The two-point cluster function $Y_2(x)=1-\langle\rho(E-x/2)\rho(E+x/2)\rangle$ gives the disconnected part of the two-point correlation function. One has the basic relation

$$\hat{y}_2(x) = Y_2\left(\frac{x}{f}\right) \quad (1)$$

that expresses the two-point cluster function of a spectrum with a fraction 1-f of missing levels in terms of the same function of the complete spectrum. We use capital letters to denote the quantities of the complete intial spectrum and small cases with a superscript for the incomplete ones. Similar scaling relations hold for higher-cluster functions. From (1) other statistical measures can be easily derived. For instance the form factor $K(\tau)=1-B(\tau)$, where $B(\tau)$ is the Fourier transform of the cluster function, transforms as follows:

$$\hat{k}(\tau) = 1 - f + fK(f\tau). \quad (2)$$

Similarly, the number variance $\hat{\sigma}^2$ (variance of the number of levels contained in an interval of length L) of the transformed spectrum is expressed in terms of the same quantity of the complete spectrum

$$\hat{\sigma}^2(L) = (1-f)L + f^2\Sigma^2\left(\frac{L}{f}\right). \quad (3)$$

The important feature of this relation is the appearance of a linear term suggesting the same behavior as for critical statistics. In particular, the Poisson expression is recovered when $f\to 0$.

Another set of statistical measures are the $E(n,s)$ functions (n-level probability functions) that give the probabilities of finding n levels ($n=0,1,2\ldots,$) inside a segment of length s. If their expressions $E(n,s)$ for the complete spectrum are known then the first ($n=0$) of these functions (gap probability function), when only a partial fraction f of levels taken at random remains, is given by

$$\hat{e}(s,f) = \sum_{k=0}^{\infty}(1-f)^k E\left(k,\frac{s}{f}\right), \quad (4)$$

which follows from the fact that 1-f is the probability that one level was dropped. By the same argument the NND $\hat{p}(s,f)$ is given by

$$\hat{p}(s,f) = \sum_{k=0}^{\infty}(1-f)^k P\left(k,\frac{s}{f}\right), \quad (5)$$

where the $P(k,s)$ are the density distributions of the spacings between two levels containing k levels inside the complete sequence [$P(0,s)$ is the NND of the complete sequence, $\hat{p}(s,f)$ the one corresponding to the incomplete sequence]. This expression was first proposed as an *anzatz* [18] and in Ref. [19] it is shown that the coefficients $f(1\text{-}f)^k$ maximizes Shannon entropy with constraints appropriately defined.

The above equations show that these expressions for the gap probability and NND of the transformed spectrum are the generating functions of all the n-level probability functions and spacing distributions of the complete spectrum. Indeed, by defining the generating function

$$G(t,z) = \sum_{k=0}^{\infty}(-1)^k(z-1)^k E(k,t) \quad (6)$$

such that

$$E(n,t) = \frac{(-1)^n}{n!}\left[\frac{\partial G(t,z)}{\partial z^n}\right]_{z=1} \quad (7)$$

then, from Eq. (4), the identification $\hat{e}(s,f)=G(\frac{s}{f},f)$ can be made. Of course, an equivalent identification also holds for the incomplete NND given by Eq. (5).

As a trivial example of the above relation, the expressions $E(n,s)=\frac{s^n}{n!}\exp(-s)$ for the Poissonian n-probability functions are generated by the function $G(t,z)=\exp(-tz)$. Taking $t=\frac{s}{f}$ and $z=f$ we can check that they are not affected by the dropping level operation.

We apply now the above formalism to initial spectra of standard RMT. By observing that the $\hat{e}$ function is the generating function of the n-level probability functions we can establish connections between the statistical properties of the transformed spectra and the RMT Fredholm determinants. These are the determinants $D(t,z)=\det(1-zK)$ and $D_\pm(t,z)=\det(1-zK_\pm)$, where K and $K_\pm$ are, respectively, the integral operators with kernels $K(x,y)=\frac{1}{\pi}\sin(x-y)/(x-y)$ and $K_\pm(x,y)=K(x,y)\pm K(x,-y)$ defined on $L^2([0,\pi s])$ [16].

Starting with the unitary Gaussian ensemble (GUE), the n-probability functions (in the sequel, the index β with $\beta=1,$, 2, and 4 denotes quantities of the orthogonal, unitary, and symplectic ensembles, respectively) are given in terms of $D(t,z)$ by

$$E_2(n,t) = \frac{(-1)^n}{n!}\left[\frac{\partial D(t,z)}{\partial z^n}\right]_{z=1}. \quad (8)$$

For GOE the even and the odd n-probability functions are given by

$$E_1(2n,t) = \sum_{k=0}^{n}E_+(k,t) - \sum_{k=0}^{n-1}E_-(k,t) \quad (9)$$

with $E_-(-1,t)=0$ and

$$E_1(2n+1,t) = \sum_{k=0}^{n}[E_-(k,t) - E_+(k,t)], \quad (10)$$

where $E_\pm$ are expressed in terms of $D_\pm$ as E_2 and D in (8). In terms of $E_\pm$, the symplectic n-probability functions are given by

$$E_4(2n,t) = \frac{1}{2}[E_+(k,2t) + E_-(k,2t)]. \quad (11)$$

On the other hand Jimbo *et al.* [20] have shown that the determinant $D(t,z)$ is given by

$$\ln D(t,z) = \int_0^{\pi t} \frac{\sigma(x,z)}{x} dx, \tag{12}$$

where $\sigma(x,z)$ is the solution of the differential equation

$$\left(x\frac{d^2\sigma}{dx^2}\right) + 4\left(x\frac{d\sigma}{dx} - \sigma\right)\left[x\frac{d\sigma}{dx} - \sigma + \left(\frac{d\sigma}{dx}\right)\right] = 0 \tag{13}$$

that satisfies the boundary condition $\sigma(x,z) \sim -\frac{z}{\pi}x$ when $x \to 0$. It has also been shown that $D_\pm(t,z)$ are given in terms of $D(t,z)$ as follows:

$$\ln D_\pm(t,z) = \frac{1}{2}\ln D(t,z) \pm \frac{1}{2}\int_0^t dx \sqrt{-\frac{d^2}{x^2}\ln D(t,z)}. \tag{14}$$

From the above we derive that the incomplete gap functions in the three cases are connected with the respective determinants $D(t,z)$ and $D_\pm(t,z)$ by the relations

$$\hat{e}_2(s,f) = D\left(\frac{s}{f}, f\right) \tag{15}$$

for the unitary,

$$\hat{e}_1\left(\frac{s}{f}, f\right) = \frac{1}{2-f}\left[D_+\left(\frac{s}{f}, 2f - f^2\right) + (1-f)D_-\left(\frac{s}{f}, 2f - f^2\right)\right] \tag{16}$$

for the orthogonal, after some algebra, and finally for the symplectic

$$\hat{e}_4\left(\frac{s}{f}, f\right) = \frac{1}{2}\left[D_+\left(\frac{2s}{f}, f\right) + D_-\left(\frac{2s}{f}, f\right)\right]. \tag{17}$$

The above equations provide exact expressions for the gap functions of both complete and randomly incomplete RMT spectra. They are one of the main results of the present communication as they provide a physical interpretation for all the real values of the parameter z in the interval $[0,1]$ appearing in the Fredholm determinant. With $z=1$ they have been used to derive asymptotics of the spacings for large separations [21]. We now show that when spectra are incomplete a major change in their asymptotics appears. This follows from the fact that with $0 < z < 1$, the function $\sigma(x,z)$ behaves, when $x \to \infty$, as follows [22]:

$$\sigma(x,z) = \frac{x}{\pi}\ln(1-z). \tag{18}$$

Substituting into the above equations, one finds that to leading order

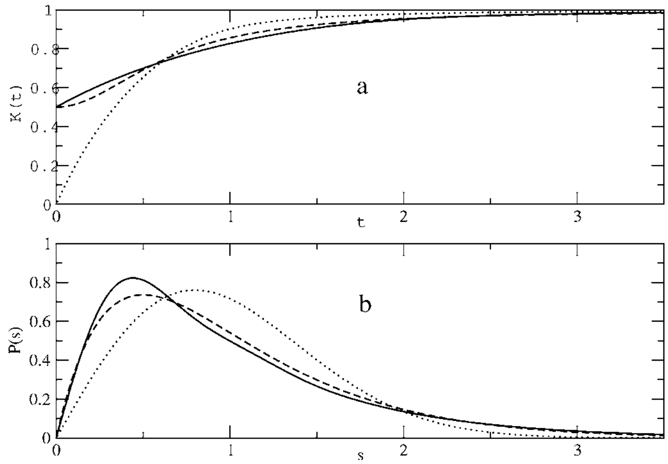

FIG. 1. (a) Form factor; (b) nearest neighbor spacing distribution (NND). Full line: incomplete ($f=1/2$) orthogonal ensemble; dashed line: the semi-Poisson model; dotted line: orthogonal ensemble.

$$\hat{e}_\beta(s,f) = \exp\left[\frac{s}{f}\ln(1-f)\right] \tag{19}$$

for $\beta = 1,2$ while f has to be multiplied by 2 for $\beta = 1$. When $f \to 0$ one gets the Poisson behavior $\exp(-s)$ irrespective of the value of β.

Let us remark that although we discuss here the $\hat{e}$ function, the identification of the incomplete NND $\hat{p}$ function as a generating function of spacing distributions can also be used to perform a similar analysis [23].

We can now compare the incomplete GOE case with the semi-Poisson model. This model gives rise to a linear number variance with slope $1/2$. By taking $f=1/2$ in (3), the incomplete GOE spectrum has the same behavior for large values of L apart from a small contribution of the Σ^2 term. Considering the form factor we have for the incomplete sequence

$$\hat{k}(\tau) = 1 - f + f\begin{cases} 2f\tau + f\tau\ln(1 + 2f\tau), & 0 \leq f\tau \leq 1, \\ 2 - f\tau\ln\left(\dfrac{2f\tau + 1}{2f\tau - 1}\right), & f\tau \geq 1, \end{cases} \tag{20}$$

and for the semi-Poisson model

$$K(\tau) = \frac{2 + \pi^2\tau^2}{4 + \pi^2\tau^2}. \tag{21}$$

In Fig. 1(a) these two functions are compared. They do not coincide but they show strong similarities: they both start with a value $1/2$ at the origin, reflecting an identical (lack of) rigidity and for large τ they both tend to 1 like $1/\tau^2$. Concerning the spacing distributions, recall first that the semi-Poisson NND is given by $4s\exp(-2s)$ [13]. Close to the origin only the $k=0$ term contributes in (5), therefore $\hat{p}(s,f)\sim P_{GOE}(0,\frac{s}{f})\sim\frac{s}{f}\frac{\pi^2}{6}$ leading with $f=1/2$ to a slightly smaller slope for the incomplete spectrum. On the other hand, taking $f=1/2$ in (19) we have for large separations a decay $\exp[-2(\ln 2)s]$ slower than the semi-Poisson one. This comparison is illustrated in Fig. 1(b). In summary, although with similarities, the incomplete GOE model presents differences with respect to the semi-Poisson model.

It is worth noticing that although our procedure interpolates between initial and final (Poisson) spectra, it gives rise to different results from the ones resulting from superposing in an uncorrelated way different spectra [16]. In this latter case, for example, level repulsion is destroyed while the dropping mechanism presented here preserves it for all values of f.

Let us now apply the dropping procedure to a picket fence spectrum defined as a sequence of points located, say, at $\cdots-3/2,-1/2,1/2,3/2\cdots$. From the definition, one can write

$$P(n,s)=\delta(s-n-1), \tag{22}$$

and

$$E(n,s)=\begin{cases}1-|s-n|, & |s-n|\leq 1,\\ 0, & |s-n|\geq 1,\end{cases} \tag{23}$$

and for the two-point cluster function

$$Y_2(x)=1-\sum_{n=0}^{\infty}\delta[x-(n+1)], \tag{24}$$

which follows from (22) and the general relation $Y_2(x)=1-\Sigma_{n=0}^{\infty}P(n,x)$. The picket fence is the most "correlated" spectrum and its rigidity reflects, for instance, in the smallness of the number variance

$$\Sigma^2(L)=L-[L]-(L-[L])^2, \tag{25}$$

where $[L]$ stands for the integer part of L.

For later comparison with the behavior of weakly confined eigenvalues, consider now what we denote as the "continuous" version of the picket fence spectrum, namely each point is randomly, independently, and uniformly distributed inside an interval of unit length around the values $\cdots-3/2,-1/2,1/2,3/2,\cdots$. In this case, the δ-functions in (22) of the spacing distributions become the "triangles"

$$P(n,s)=\begin{cases}0, & s\leq n,\\ s-n, & n\leq s\leq n+1,\\ s-n-2, & n+1\leq s\leq n+2,\\ 0, & n+2\leq s,\end{cases} \tag{26}$$

that preserve the normalization $\langle 1\rangle=1$ and $\langle s\rangle=n+1$. From (26) we find that the cluster function takes the simple expression

$$Y_2(x)=\begin{cases}1-x, & 0\leq x\leq 1,\\ 0, & 1\leq x,\end{cases} \tag{27}$$

from which follows

$$K(\tau)=1-\left[\frac{\sin(\pi\tau)}{\pi\tau}\right]^2, \tag{28}$$

and

$$\Sigma^2(L)=\begin{cases}L-L^2+L^3/3, & 0\leq L\leq 1,\\ 1/3, & 1\leq L.\end{cases} \tag{29}$$

Using the general relations for incomplete spectra we find for the transformed cluster function

$$\hat{y}_2(x)=\begin{cases}1-x/f, & 0\leq x\leq f,\\ 0, & f\leq x,\end{cases} \tag{30}$$

form factor

$$\hat{k}(\tau)=1-f\left[\frac{\sin(f\pi\tau)}{f\pi\tau}\right]^2 \tag{31}$$

and number variance

$$\hat{\sigma}^2(L)=(1-f)L+f^2\begin{cases}L/f-(L/f)^2+(L/f)^3/3, & 0\leq L\leq f,\\ 1/3, & f\leq L.\end{cases} \tag{32}$$

Finally, substituting (26) into (5), the NND for the incomplete sequence becomes

$$\hat{p}(s,f)=\begin{cases}f^{-1}s, & 0\leq s\leq f,\\ (1-f)^{n-1}(1+nf-s), & nf\leq s\leq(n+1)f,\ n=1,2,3\ldots.\end{cases} \tag{33}$$

(see Fig. 2 for illustration).

Let us compare these results with the behavior of the eigenvalues of matrix ensembles in which the eigenvalues are confined by a weak logarithm-normal potential. By that is meant eigenvalues of $N\times N$ random matrices M distributed according to

$$P(M)=\exp[-\mathrm{Tr}V(M)] \tag{34}$$

with $V(x)=\ln^2|x|/\gamma$. It has been shown in Ref. [9] that in the limit of weak confinement $\gamma\to\infty$, the spectrum, whose N levels tend to locate around the sites of a crystal lattice, has, after unfolding, the following structure. Construct $2N$ intervals of length $1/2$ symmetrically with respect to the origin. The N levels occupy randomly these intervals as follows: (i) in an interval there is at most one level, (ii) intervals symmetric with respect to the origin cannot be simultaneously occupied. This last property introduces long range correlations and lack of stationarity (translation invariance). By performing an average over the spectrum or, equivalently, by considering only the first (or second) half of the spectrum, the effect of this long range correlation is washed out and one obtains, for the two-point cluster function

$$Y_2(x) = \begin{cases} 1 - 2x, & 0 \leq x \leq 1/2, \\ 0, & 1/2 \leq x, \end{cases} \tag{35}$$

and for the NND (see Ref. [9])

$$P(s) = \begin{cases} 2s, & 0 \leq s \leq 1/2, \\ 2^{-n+1}(1 + n/2 - s), & n/2 \leq s \leq (n+1)/2, \quad n = 1,2,3 \ldots . \end{cases} \tag{36}$$

Obviously, (35) and (36) are just (30) and (33) with $f=1/2$, corresponding to the continuous picket fence for which half of the levels have been randomly removed (see Fig. 2).

Let us finally mention an example of a different sort of mechanism than dropping or removing points. Here points are located on a line in the plane and, as a parameter is varied, a fraction of them leave the line. Specifically, consider the roots of a random polynomial of degree N,

$$P_N(z) = \sum_{k=0}^{N} a_k z^k, \tag{37}$$

where the a_k's are independent Gaussian complex random variables (real and imaginary parts centered at zero and variance σ^2) [24]. If one imposes the symmetry [self-inversive (SI) symmetry, also called self-reciprocal or conjugate reciprocal]

$$a_{N-k} = \bar{a}_k, \tag{38}$$

where the bar denotes complex conjugate, one can see that the roots of $P_N(z)$ lie either on the unit circle C or appear in pairs symmetrically located under inversion with respect to it. The relevant parameter of the model is $\epsilon=\sigma\sqrt{N}$. As ϵ increases, some roots leave C and in the limit $\epsilon\to\infty$, on the average, a fraction $\phi=1/\sqrt{3}$ of the roots remains on it [24,25]. On the other hand, in Ref. [26], the restricted class of SI polynomials having all the roots on C has been considered. It has been found that their statistical properties coincide with those of eigenvalues of the orthogonal ensemble (OE) of random matrices ($\beta=1$). One may then ask whether properties of the unrestricted class of SI polynomials, in the limit $\epsilon\to\infty$, share some properties with the ones corresponding to dropping at random a fraction $1-\phi$ of zeros of the restricted polynomials. Consider, for instance, the NND. In both cases it starts linearly at the origin, with a slope $\pi^2/6$ for the restricted case and $\pi^2/(10\sqrt{3})$ for the unrestricted case [24], which is five times smaller than what would result from randomly dropping a fraction $1-\phi$ of points from an OE sequence, namely $\pi^2(6\phi)^{-1}$ [see Eq. (5)]. Leaving C does not have the same effect as randomly dropping points on it. The points who move and locate on the complex plane "interact" with those remaining on C.

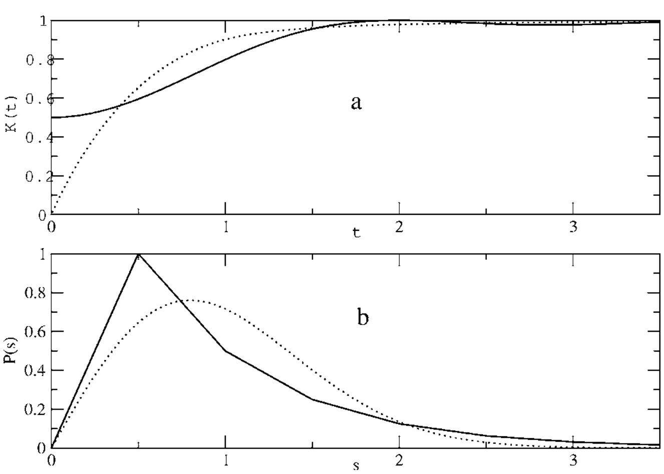

FIG. 2. (a) Form factor; (b) nearest neighbor spacing distribution (NND). Full line: incomplete ($f=1/2$) continuous picket fence; dotted line: orthogonal ensemble.

In conclusion, by dropping at random a fraction of levels of a given spectrum a family of spectra is generated. Its statistical properties are intermediate between those of the initial one and a Poisson spectrum. Applied to eigenvalues of RMT, the family contains as a particular case a model exhibiting some of the features of critical intermediate statistics. Fredholm determinants of argument z are one of the basic structures appearing in RMT (one is usully interested in properties corresponding to $z=1$). We show, for $z<1$, that properties of Fredholm determinants also characterize the behavior of incomplete spectra and that z corresponds to the remaining fraction of levels. When the dropping procedure is applied to a picket fence spectrum we show that the generated family has the statistical properties of an ensemble of eigenvalues weakly confined. Finally we compare with a system for which, when a parameter is varied, a fraction of points on a line are not dropped but move on the complex plane.

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